

# Hyperbolic Spaces and Ptolemy Möbius Structures

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## Abstract

We characterize the class of Gromov hyperbolic spaces, whose boundary at infinity allow canonical Möbius structures.

## 1 Introduction

There is a deep and well studied relation between the geometry of the classical hyperbolic space and the Möbius geometry of its boundary at infinity. This relation can be generalized in a nice way to  $\text{CAT}(-1)$  spaces.

Let  $X$  be a  $\text{CAT}(-1)$  space with boundary  $Z = \partial_\infty X$ . For every basepoint  $o \in X$  one can define the Bourdon metric  $\rho_o(x, y) = e^{-(x|y)_o}$  on  $Z$ , where  $(\cdot | \cdot)_o$  is the Gromov product with respect to  $o$ , compare [B1]. For different basepoints  $o, o' \in X$  the metrics  $\rho_o, \rho_{o'}$  are Möbius equivalent and thus define a Möbius structure on  $Z$ . By [FS1] this Möbius structure is ptolemaic.

On the other hand, examples show that not every ptolemaic Möbius structure arises as boundary of a  $\text{CAT}(-1)$  space. In this paper we enlarge the class of  $\text{CAT}(-1)$  spaces in a way that this larger class corresponds exactly to the spaces which have a ptolemaic Möbius structure at infinity.

*Definition 1.1.* A metric space is called *asymptotically*  $\text{PT}_{-1}$ , if there exists some  $\delta > 0$  such that for all quadruples  $x_1, x_2, x_3, x_4 \in X$  we have

$$e^{\frac{1}{2}(\rho_{1,3} + \rho_{2,4})} \leq e^{\frac{1}{2}(\rho_{1,2} + \rho_{3,4})} + e^{\frac{1}{2}(\rho_{1,4} + \rho_{2,3})} + \delta e^{\frac{1}{2}\rho},$$

where  $\rho_{i,j} = d(x_i, x_j)$  and  $\rho = \max_{i,j} \rho_{i,j}$ .

We discuss this curvature condition in more detail later and compare it in section 3.3 with the asymptotically  $\text{CAT}(-1)$  condition, which is formulated in more familiar comparison terms. It turns out that  $\text{CAT}(-1)$  are asymptotically  $\text{PT}_{-1}$  and that the relation between these spaces and the Möbius geometry of their boundaries can be expressed in the following two results:

**Theorem 1.2.** *Let  $X$  be asymptotically  $\text{PT}_{-1}$ , then  $X$  is a boundary continuous Gromov hyperbolic space. For every basepoint  $o \in X$ ,  $\rho_o(x, y) = e^{-(x|y)_o}$  defines a metric on  $\partial_\infty X$ . For different basepoints these metrics are Möbius equivalent and thus define a canonical Möbius structure  $\mathcal{M}$  on  $\partial_\infty X$ . The Möbius structure  $\mathcal{M}$  is complete and ptolemaic.*

**Theorem 1.3.** *Let  $(Z, \mathcal{M})$  be a complete and ptolemaic Möbius space. Then there exists an asymptotically  $\text{PT}_{-1}$  space  $X$  such that  $\partial_\infty X$  with its canonical Möbius structure is Möbius equivalent to  $(Z, \mathcal{M})$ .*

The results can be viewed as a characterization of the class of Gromov hyperbolic spaces, whose boundary allows a canonical Möbius structure.

In the proof we use a hyperbolic cone construction due to Bonk and Schramm [BoS], which associates to a metric space  $(Z, d)$  a cone  $(\text{Con}(Z), \rho)$ . We show in Proposition 4.1 that if  $(Z, d)$  is ptolemaic, then the cone is asymptotically  $\text{PT}_{-1}$ . This method can also be used to obtain a characterization of visual Gromov hyperbolic spaces in the spirit of the [BoS]. Recall that two metric spaces  $X$  and  $Y$  are roughly similar, if there are constants  $K, \lambda > 0$  and a map  $f : X \rightarrow Y$  such that for all  $x, y \in X$

$$|\lambda d_X(x, y) - d_Y(f(x), f(y))| \leq K$$

and in addition  $\sup_{y \in Y} d_Y(y, f(X)) \leq K$ .

A theorem of Bonk and Schramm states that a visual Gromov hyperbolic space with doubling boundary is roughly similar to a convex subset of the real hyperbolic space  $\mathbb{H}^n$  for some integer  $n$ .

We have a version without conditions on the boundary:

**Theorem 1.4.** *Every visual Gromov hyperbolic space is roughly similar to some asymptotically  $\text{PT}_{-1}$  space.*

We discuss some open questions in Remark 3.2. The structure of the paper is as follows. In section 2 we recall the basic facts about metric Möbius geometry and boundary continuous Gromov hyperbolic spaces. In section 3 we introduce the  $\text{PT}_\kappa$  property, discuss asymptotically  $\text{PT}_{-1}$  spaces and prove Theorem 1.2. In section 4 we introduce hyperbolic cones and prove Theorem 1.3. The proof of Theorem 1.4 is in section 5.

## 2 Preliminaries

### 2.1 Möbius Structures

Let  $Z$  be a set which contains at least two points. An *extended metric* on  $Z$  is a map  $d : Z \times Z \rightarrow [0, \infty]$ , such that there exists a set  $\Omega(d) \subset Z$  with cardinality  $\#\Omega(d) \in \{0, 1\}$ , such that  $d$  restricted to the set  $Z \setminus \Omega(d)$  is a metric (taking only values in  $[0, \infty)$ ) and such that  $d(z, \omega) = \infty$  for all

$z \in Z \setminus \Omega(d)$ ,  $\omega \in \Omega(d)$ . Furthermore  $d(\omega, \omega) = 0$ .

If  $\Omega(d)$  is not empty, we call the unique  $\omega \in \Omega(d)$  simply *the point at infinity* of  $(Z, d)$ . We write  $Z_\omega$  for the set  $Z \setminus \{\omega\}$ .

The topology considered on  $(Z, d)$  is the topology with the basis consisting of all open distance balls  $B_r(z)$  around points in  $z \in Z_\omega$  and the complements  $D^C$  of all closed distance balls  $D = \overline{B}_r(z)$ .

We call an extended metric space  $(Z, d)$  *complete*, if first every Cauchy sequence in  $Z_\omega$  converges and secondly if the infinitely remote point  $\omega$  exists in case that  $Z_\omega$  is unbounded. For example the real line  $(\mathbb{R}, d)$ , with its standard metric is *not* complete (as extended metric space), while  $(\mathbb{R} \cup \{\infty\}, d)$  is complete.

We say that a quadruple  $(x, y, z, w) \in Z^4$  is *admissible*, if no entry occurs three or four times in the quadruple. We denote with  $Q \subset Z^4$  the set of admissible quadruples. We define the *cross ratio triple* as the map  $\text{crt} : Q \rightarrow \Sigma \subset \mathbb{R}P^2$  which maps admissible quadruples to points in the real projective plane defined by

$$\text{crt}(x, y, z, w) = (d(x, y)d(z, w) : d(x, z)d(y, w) : d(x, w)d(y, z)),$$

here  $\Sigma$  is the subset of points  $(a : b : c) \in \mathbb{R}P^2$ , where all entries  $a, b, c$  are nonnegative or all entries are non positive.

We use the standard conventions for the calculation with  $\infty$ . If  $\infty$  occurs once in  $Q$ , say  $w = \infty$ , then  $\text{crt}(x, y, z, \infty) = (d(x, y) : d(x, z) : d(y, z))$ . If  $\infty$  occurs twice, say  $z = w = \infty$  then  $\text{crt}(x, y, \infty, \infty) = (0 : 1 : 1)$ .

Similar as for the classical cross ratio there are six possible definitions by permuting the entries and we choose the above one.

A map  $f : Z \rightarrow Z'$  between two extended metric spaces is called *Möbius*, if  $f$  is injective and for all admissible quadruples  $(x, y, z, w)$  of  $X$ ,

$$\text{crt}(f(x), f(y), f(z), f(w)) = \text{crt}(x, y, z, w).$$

Möbius maps are continuous.

Two extended metric spaces  $(Z, d)$  and  $(Z, d')$  are *Möbius equivalent*, if there exists a bijective Möbius map  $f : Z \rightarrow Z$ . In this case also  $f^{-1}$  is a Möbius map and  $f$  is in particular a homeomorphism.

We say that two extended metrics  $d$  and  $d'$  on the same set  $Z$  are *Möbius equivalent*, if the identity map  $\text{id} : (Z, d) \rightarrow (Z, d')$  is a Möbius map. Möbius equivalent metrics define the same topology on  $Z$ . It is also not difficult to check that for Möbius equivalent metrics  $d$  and  $d'$  the space  $(Z, d)$  is complete if and only if  $(Z, d')$  is complete.

The Möbius equivalence of metrics of metrics on a given set  $Z$  is clearly an equivalence relation. A *Möbius structure*  $\mathcal{M}$  on  $Z$  is an equivalence class of extended metrics on  $Z$ .

A pair  $(Z, \mathcal{M})$  of a set  $Z$  together with a Möbius structure  $\mathcal{M}$  on  $Z$  is called a *Möbius space*. A Möbius structure well defines a topology on  $Z$ , thus a Möbius space is in particular a topological space. Since completeness is also a Möbius invariant we can speak about *complete* Möbius structures.

In general two metrics in  $\mathcal{M}$  can look very different. However if two metrics have the same remote point at infinity, then they are homothetic (see [FS2]):

**Lemma 2.1.** *Let  $\mathcal{M}$  be a Möbius structure on a set  $X$ , and let  $d, d' \in \mathcal{M}$ , such that  $\omega \in X$  is the remote point of  $d$  and of  $d'$ . Then there exists  $\lambda > 0$ , such that  $d'(x, y) = \lambda d(x, y)$  for all  $x, y \in X$ .*

An extended metric space  $(Z, d)$  is called a *Ptolemy space*, if for all quadruples of points  $\{x, y, z, w\} \in Z^4$  the *Ptolemy inequality* holds

$$d(x, y) d(z, w) \leq d(x, z) d(y, w) + d(x, w) d(y, z)$$

We can reformulate this condition in terms of the cross ratio triple. Let  $\Delta \subset \Sigma$  be the set of points  $(a : b : c) \in \Sigma$ , such that the entries  $a, b, c$  satisfy the triangle inequality. This is obviously well defined.

Then an extended space is Ptolemy, if  $\text{crt}(x, y, z, w) \in \Delta$  for all allowed quadruples  $Q$ .

This description shows that the Ptolemy property is Möbius invariant and thus a property of the Möbius structure  $\mathcal{M}$ .

The importance of the Ptolemy property comes from the following fact (see e.g. [FS2]):

**Theorem 2.2.** *A Möbius structure  $\mathcal{M}$  on a set  $Z$  is Ptolemy, if and only if for all  $\omega \in Z$  there exists  $d_\omega \in \mathcal{M}$  with  $\Omega(d_\omega) = \{\omega\}$ .*

The metric  $d_\omega$  can be obtained by metric involution. If  $d$  is a metric on  $Z$  then

$$d_\omega(z, z') = \frac{d(z, z')}{d(\omega, z)d(\omega, z')}$$

gives the required metric.

## 2.2 Boundary continuous Gromov hyperbolic spaces

We recall some basic facts from the theory of Gromov hyperbolic spaces, compare e.g [BS]

A metric space  $(X, d)$  is called *Gromov hyperbolic* if there exists some  $\delta > 0$  such that for all quadruples  $x_1, x_2, x_3, x_4 \in X$  we have

$$\rho_{1,3} + \rho_{2,4} \leq \max\{\rho_{1,2} + \rho_{3,4}, \rho_{1,4} + \rho_{2,3}\} + \delta,$$

where  $\rho_{i,j} = d(x_i, x_j)$ .

For three points  $x, y, z \in X$  one defines the *Gromov product*

$$(x|y)_z = \frac{1}{2} (|zx| + |zy| - |xy|) ,$$

where we write  $|xy|$  as a short version of  $d(x, y)$ .

A sequence  $(x_i)$  *converges at infinity*, if for some (and hence every) basepoint  $o \in X$  we have  $\lim_{i,j \rightarrow \infty} (x_i|x_j)_o = \infty$ . Two such sequences  $(x_i), (y_i)$  are called *equivalent*, if  $\lim (x_i|y_i)_o = \infty$ . The boundary  $\partial_\infty X$  consist of the equivalence classes of these sequences.

For two points  $\zeta, \xi \in \partial_\infty X$  and a base point  $o \in X$  one defines

$$(\zeta|\xi)_o = \inf_{i \rightarrow \infty} \liminf (x_i|y_i)_o \quad (1)$$

where the infimum is taken over all sequences  $(x_i) \in \zeta$  and  $(y_i) \in \xi$ . In a similar way we also define  $(x|\xi)_o$ , where  $o, x \in X$  and  $\xi \in \partial_\infty X$ .

We remark that the sequence  $(x_i|y_i)_o$  does not necessarily converge, therefore we need the complicated definition in (1).

A Gromov hyperbolic space is called *boundary continuous*, if the Gromov product extends continuously to the boundary in the following way: if  $(x_i), (y_i)$  are sequences in  $X$  which converge to points  $\bar{x}, \bar{y}$  in  $X$  or  $\partial_\infty X$ , then  $(x_i|y_i)_o \rightarrow (\bar{x}|\bar{y})_o$  for all base points  $o \in X$ . For boundary continuous spaces one can define nicely Busemann functions. If  $\omega \in \partial_\infty X$  and  $o \in X$  a base point, then

$$b_{\omega,o}(x) = \lim_{i \rightarrow \infty} (|xw_i| - |w_i o|) \quad (2)$$

where  $w_i \rightarrow \omega$  is the Busemannfunction of  $\omega$  normalized to have the value 0 at the point  $o \in X$ . We have the formula:

$$b_{\omega,o}(x) = (\omega|o)_x - (\omega|x)_o \quad (3)$$

We also define form  $\omega \in \partial_\infty X$  a base point  $o \in X$  and  $x, y, z$  from  $X$  or  $\partial_\infty X \setminus \{\omega\}$

$$(x|y)_{\omega,o} = (x|y)_o - (\omega|x)_o - (\omega|y)_o.$$

### 3 Asymptotic $\text{PT}_\kappa$ spaces

In subsection 3.1 we define general  $\text{PT}_\kappa$  spaces. Then in section 3.2 we introduce the more general notion of asymptotic  $\text{PT}_\kappa$  spaces and compare it in section 3.3 with the notion of asymptotically  $\text{CAT}(\kappa)$  spaces. Finally in section 3.4 we prove Theorem 1.2.

### 3.1 The $PT_\kappa$ inequality

A metric space  $(X, d)$  is called a  $PT_\kappa$ -space, if for points  $x_1, x_2, x_3, x_4 \in X$ , we have

$$\operatorname{sn}_\kappa\left(\frac{\rho_{1,3}}{2}\right) \operatorname{sn}_\kappa\left(\frac{\rho_{2,4}}{2}\right) \leq \operatorname{sn}_\kappa\left(\frac{\rho_{1,2}}{2}\right) \operatorname{sn}_\kappa\left(\frac{\rho_{3,4}}{2}\right) + \operatorname{sn}_\kappa\left(\frac{\rho_{1,4}}{2}\right) \operatorname{sn}_\kappa\left(\frac{\rho_{2,3}}{2}\right) \quad (4)$$

where  $\rho_{i,j} = d(x_i, x_j)$  and  $\operatorname{sn}_\kappa$  is the function

$$\operatorname{sn}_\kappa(x) = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}x) & \text{if } \kappa > 0, \\ x & \text{if } \kappa = 0, \\ \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa}x) & \text{if } \kappa < 0. \end{cases}$$

In the case that  $\kappa > 0$  we assume in addition that the diameter is bounded by  $\frac{\pi}{\sqrt{\kappa}}$ .

It is well known that the standard space forms  $M_\kappa^n$  of constant curvature  $\kappa$  satisfy the  $PT_\kappa$  inequality. For the euclidean space this is the classical ptolemaic inequality and for the other spaces it is proved in [H]. By comparison we obtain the result also for  $CAT(\kappa)$ -spaces.

**Proposition 3.1.** *Every  $CAT(\kappa)$  space satisfies the  $PT_\kappa$  inequality.*

*Proof.* A  $CAT(\kappa)$  spaces,  $\kappa \in \mathbb{R}$ , can be characterized by a 4-point condition, [BH, Proposition 1.11]. Suppose  $x_i \in X$  for  $0 \leq i \leq 4$ , with  $x_0 = x_4$ , and  $x_0 = x_4$ , there exist four points  $\bar{x}_i \in M_\kappa^2$  with  $\bar{x}_0 = \bar{x}_4$  such that

$$d(x_i, x_{i-1}) = |\bar{x}_i - \bar{x}_{i-1}|, 1 \leq i \leq 4,$$

$$d(x_1, x_3) \leq |\bar{x}_1 - \bar{x}_3| \quad \text{and} \quad d(x_2, x_4) \leq |\bar{x}_2 - \bar{x}_4|.$$

Since  $M_\kappa^2$  satisfy the  $PT_\kappa$  inequality the result follows.  $\square$

*Remark 3.2.* The following questions arises naturally: is a geodesic  $PT_\kappa$  space  $CAT(\kappa)$ ? A positive answer would imply a nice four point characterization of  $CAT(\kappa)$  spaces. In the case  $\kappa = 0$  this is not true (see [FLS]), but the counterexamples are not locally compact and there are partial positive results in the locally compact case (see e.g. [BuFW], [MS]). For  $\kappa < 0$  the question is completely open.

### 3.2 Asymptotic $PT_\kappa$ inequality for $\kappa < 0$

One obtains the asymptotic  $PT_\kappa$  property (for  $\kappa < 0$ ) by weakening equation the  $PT_\kappa$  inequality and allowing some error term. Instead of equation (4) we require that for some universal  $\delta \geq 0$  we have

$$\begin{aligned} \operatorname{sn}_\kappa\left(\frac{\rho_{1,3}}{2}\right) \operatorname{sn}_\kappa\left(\frac{\rho_{2,4}}{2}\right) &\leq \\ \operatorname{sn}_\kappa\left(\frac{\rho_{1,2}}{2}\right) \operatorname{sn}_\kappa\left(\frac{\rho_{3,4}}{2}\right) &+ \operatorname{sn}_\kappa\left(\frac{\rho_{1,4}}{2}\right) \operatorname{sn}_\kappa\left(\frac{\rho_{2,3}}{2}\right) + \delta e^{\frac{\sqrt{-\kappa}}{2}\rho} \end{aligned}$$

It is more convenient to formulate this condition using only exponential functions. It is easy to check that these conditions are equivalent.

**Definition 3.3.** A metric space is called asymptotic  $PT_\kappa$  for some  $\kappa < 0$ , if there exists some  $\delta \geq 0$  such that for all quadruples  $x_1, x_2, x_3, x_4 \in X$  we have

$$e^{\frac{\sqrt{-\kappa}}{2}(\rho_{1,3}+\rho_{2,4})} \leq e^{\frac{\sqrt{-\kappa}}{2}(\rho_{1,2}+\rho_{3,4})} + e^{\frac{\sqrt{-\kappa}}{2}(\rho_{1,4}+\rho_{2,3})} + \delta e^{\frac{\sqrt{-\kappa}}{2}\rho}$$

Here  $\rho_{i,j} = d(x_i, x_j)$  and  $\rho = \max_{i,j} \rho_{i,j}$ .

**Remark 3.4.** The asymptotic  $PT_\kappa$  condition is a strong curvature condition. It implies e.g. that  $X$  does not contain flat strips: if a space contains a flat strip of width  $a > 0$ , then it contains quadruples with  $\rho_{1,3} = \rho_{2,4} = \sqrt{t^2 + a^2}$ ,  $\rho_{1,2} = \rho_{3,4} = t$  and  $\rho_{2,3} = \rho_{1,4} = a$ . These quadruples do not satisfy the asymptotic  $PT_\kappa$  inequality for fixed  $\kappa < 0$ ,  $\delta \geq 0$  and  $t \rightarrow \infty$ .

**Proposition 3.5.** Let  $0 > \kappa' > \kappa$ . If  $X$  is asymptotic  $PT_\kappa$ , then  $X$  is asymptotic  $PT_{\kappa'}$ .

*Proof.* From the asymptotic  $PT_\kappa$  inequality, we obtain that

$$e^{\frac{\sqrt{-\kappa}}{2}(\rho_{1,3}+\rho_{2,4})} \leq e^{\frac{\sqrt{-\kappa}}{2}(\rho_{1,2}+\rho_{3,4})} + e^{\frac{\sqrt{-\kappa}}{2}(\rho_{1,4}+\rho_{2,3})} + \delta e^{\frac{\sqrt{-\kappa}}{2}\rho}$$

Here  $\rho = \max_{i,j} \rho_{i,j}$  for  $i, j = 1, 2, 3, 4$ .

Since we know that for  $0 \leq x \leq 1$

$$(a+b)^x \leq a^x + b^x, a > 0, b > 0.$$

Hence

$$\begin{aligned} e^{\frac{\sqrt{-\kappa'}}{2}(\rho_{1,3}+\rho_{2,4})} &= \left(e^{\frac{\sqrt{-\kappa}}{2}(\rho_{1,3}+\rho_{2,4})}\right)^{\sqrt{\frac{-\kappa'}{-\kappa}}} \\ &\leq \left(e^{\frac{\sqrt{-\kappa}}{2}(\rho_{1,2}+\rho_{3,4})} + e^{\frac{\sqrt{-\kappa}}{2}(\rho_{1,4}+\rho_{2,3})} + \delta e^{\frac{\sqrt{-\kappa}}{2}\rho}\right)^{\sqrt{\frac{-\kappa'}{-\kappa}}} \\ &\leq e^{\frac{\sqrt{-\kappa'}}{2}(\rho_{1,2}+\rho_{3,4})} + e^{\frac{\sqrt{-\kappa'}}{2}(\rho_{1,4}+\rho_{2,3})} + \delta' e^{\frac{\sqrt{-\kappa'}}{2}\rho} \end{aligned}$$

It satisfies the asymptotic  $PT_{\kappa'}$  inequality.  $\square$

By scaling an asymptotic  $PT_\kappa$  space with the factor  $\frac{1}{\sqrt{-\kappa}}$  we obtain an asymptotic  $PT_{-1}$  space. Therefore we will discuss in the sequel mainly  $PT_{-1}$  spaces.

### 3.3 Asymptotically $\text{CAT}(\kappa)$ spaces

We relate the asymptotic  $\text{PT}(\kappa)$  property to some condition which is formulated in familiar comparison terms.

*Definition 3.6.* Let  $\kappa < 0$ . A metric space  $(X, d)$  is called *asymptotically*  $\text{CAT}(\kappa)$ , if there exists some  $\delta \geq 0$  such that for every quadruple of points  $x_1, x_2, x_3, x_4$  in  $X$  there are comparison points  $\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4$  in  $M_\kappa^2$  such that

$$\begin{aligned} d(x_1, x_2) &= |\bar{x}_1 \bar{x}_2|, & d(x_2, x_3) &= |\bar{x}_2 \bar{x}_3|, \\ d(x_3, x_4) &= |\bar{x}_3 \bar{x}_4|, & d(x_4, x_1) &= |\bar{x}_4 \bar{x}_1|, \\ d(x_1, x_3) &\leq |\bar{x}_1 \bar{x}_3|, \\ \text{sn}_\kappa\left(\frac{d(x_2, x_4)}{2}\right) &\leq \text{sn}_\kappa\left(\frac{|\bar{x}_2 \bar{x}_4|}{2}\right) + \delta. \end{aligned}$$

*Remark 3.7.* This definition makes also sense for  $\kappa = 0$ , then the last inequality is just  $d(x_2, x_4) \leq |\bar{x}_2 \bar{x}_4| + 2\delta$ . Then the condition is the rough  $\text{CAT}(0)$  condition of [BuF]. In general, if one replaces the last condition (also for  $\kappa < 0$ ) simply by the condition  $d(x_2, x_4) \leq |\bar{x}_2 \bar{x}_4| + \delta$ , then one obtains, what is called rough  $\text{CAT}(\kappa)$  in [BuF]. For  $\kappa < 0$  this is equivalent to Gromov hyperbolicity and brings no new information

**Lemma 3.8.** *If  $(X, d)$  is asymptotically  $\text{CAT}(\kappa)$ , then it is also asymptotically  $\text{PT}_\kappa$ .*

*Proof.* Let  $X$  be asymptotically  $\text{CAT}(\kappa)$  with constant  $\delta$ . Let  $x_1, x_2, x_3, x_4 \in X$  be given and let  $\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4 \in M_\kappa^2$  be comparison points according to the asymptotic  $\text{CAT}(\kappa)$  property. Let  $\rho_{i,j} = d(x_i, x_j)$ ,  $\bar{\rho}_{i,j} = |\bar{x}_i \bar{x}_j|$  and  $\rho = \max \rho_{i,j}$ . Then

$$\begin{aligned} \text{sn}_\kappa\left(\frac{\rho_{1,3}}{2}\right) \text{sn}_\kappa\left(\frac{\rho_{2,4}}{2}\right) &- \frac{\delta}{\sqrt{-\kappa}} e^{\frac{\sqrt{-\kappa}}{2}\rho} \\ &\leq \text{sn}_\kappa\left(\frac{\rho_{1,3}}{2}\right) (\text{sn}_\kappa\left(\frac{\rho_{2,4}}{2}\right) - \delta) \\ &\leq \text{sn}_\kappa\left(\frac{\bar{\rho}_{1,3}}{2}\right) \text{sn}_\kappa\left(\frac{\bar{\rho}_{2,4}}{2}\right) \\ &\leq \text{sn}_\kappa\left(\frac{\bar{\rho}_{1,2}}{2}\right) \text{sn}_\kappa\left(\frac{\bar{\rho}_{3,4}}{2}\right) + \text{sn}_\kappa\left(\frac{\bar{\rho}_{1,4}}{2}\right) \text{sn}_\kappa\left(\frac{\bar{\rho}_{2,3}}{2}\right) \\ &= \text{sn}_\kappa\left(\frac{\rho_{1,2}}{2}\right) \text{sn}_\kappa\left(\frac{\rho_{3,4}}{2}\right) + \text{sn}_\kappa\left(\frac{\rho_{1,4}}{2}\right) \text{sn}_\kappa\left(\frac{\rho_{2,3}}{2}\right). \end{aligned}$$

Thus  $X$  is asymptotically  $\text{PT}_\kappa$  with constant  $\frac{\delta}{\sqrt{-\kappa}}$ . □

### 3.4 Properties of asymptotically $\text{PT}_{-1}$ spaces

**Proposition 3.9.** *An asymptotic  $\text{PT}_{-1}$  metric space is a Gromov hyperbolic space.*



*Proof.* The asymptotic  $PT_{-1}$  inequality is

$$e^{\frac{1}{2}(\rho_{1,3}+\rho_{2,4})} \leq e^{\frac{1}{2}(\rho_{1,2}+\rho_{3,4})} + e^{\frac{1}{2}(\rho_{1,4}+\rho_{2,3})} + \delta e^{\frac{1}{2}\rho}$$

Using the triangle inequality, we see

$$\rho \leq \max\{\rho_{1,2} + \rho_{3,4}, \rho_{1,4} + \rho_{2,3}\}$$

which then implies

$$e^{\frac{1}{2}(\rho_{1,3}+\rho_{2,4})} \leq (\delta + 1)(e^{\frac{1}{2}(\rho_{1,2}+\rho_{3,4})} + e^{\frac{1}{2}(\rho_{1,4}+\rho_{2,3})})$$

and hence

$$\rho_{1,3} + \rho_{2,4} \leq \max\{\rho_{1,2} + \rho_{3,4}, \rho_{1,4} + \rho_{2,3}\} + \delta'.$$

Thus  $X$  is a Gromov hyperbolic space. □

**Lemma 3.10.** *Let  $X$  be an asymptotic  $PT_{-1}$  space. Let  $(x_i)$ ,  $(x'_i)$  and  $(y_i)$  be sequences in  $X$  satisfying*

$$\lim_{i \rightarrow \infty} (x_i | x'_i)_o = \infty, \quad \lim_{i \rightarrow \infty} (x_i | y_i)_o = a, \quad o \in X.$$

*Then*

$$\lim_{i \rightarrow \infty} (x'_i | y_i)_o = a$$

*Proof.* From the asymptotic  $PT_{-1}$  inequality, we obtain

$$\begin{aligned} e^{\frac{1}{2}(|x'_i y_i| + |ox_i|)} - e^{\frac{1}{2}(|oy_i| + |x_i x'_i|)} - \delta e^{\frac{1}{2}\rho_i} &\leq e^{\frac{1}{2}(|ox'_i| + |x_i y_i|)} \\ &\leq e^{\frac{1}{2}(|oy_i| + |x_i x'_i|)} + e^{\frac{1}{2}(|x'_i y_i| + |ox_i|)} + \delta e^{\frac{1}{2}\rho_i}, \end{aligned}$$

where  $\rho_i = \max\{|ox_i|, |ox'_i|, |oy_i|, |x_i x'_i|, |x_i y_i|, |x'_i y_i|\}$ .

Dividing both sides by  $e^{\frac{1}{2}(|ox_i| + |ox'_i| + |oy_i|)}$ , we obtain

$$e^{-(x'_i | y_i)_o} - e^{-(x_i | x'_i)_o} - E_i \leq e^{-(x_i | y_i)_o} \leq e^{-(x'_i | y_i)_o} + e^{-(x_i | x'_i)_o} + E_i,$$

where  $E_i = \delta e^{\frac{1}{2}(\rho_i - |ox_i| - |ox'_i| - |oy_i|)}$ . Note that by triangle inequalities

$$|ox_i| + |ox'_i| + |oy_i| - \rho_i \geq \min\{|ox_i|, |ox'_i|, 2(x_i | x'_i)_o\},$$

and hence  $E_i \rightarrow 0$  by our assumptions. Taking the limit, we obtain

$$\lim_{i \rightarrow \infty} (x'_i | y_i)_o = \lim_{i \rightarrow \infty} (x_i | y_i)_o = a.$$

□

As an immediate consequence we get

**Corollary 3.11.** *An asymptotic  $PT_{-1}$  space is boundary continuous.*

**Theorem 3.12.** *Let  $X$  be an asymptotic  $PT_{-1}$  metric space and  $o \in X$ , then*

$$\rho_o(x, y) = e^{-(x|y)_o}, \quad x, y \in \partial_\infty X$$

*is a metric on  $\partial_\infty X$  which is  $PT_0$ .*

*Proof.* First, we show that  $\rho_o$  is a metric on  $\partial_\infty X$ . For given three points  $x, y, z \in \partial_\infty X$ , choose sequences  $(x_i) \in x, (y_i) \in y, (z_i) \in z$ . By boundary continuity we have  $(x|z)_o = \lim_{i \rightarrow \infty} (x_i|z_i)_o$ . Then

$$e^{-(x|z)_o} = \lim_{i \rightarrow \infty} e^{\frac{1}{2}(|x_i z_i| - |x_i o| - |z_i o|)} = \lim_{i \rightarrow \infty} e^{-\frac{1}{2}(|x_i o| + |y_i o| + |z_i o|)} e^{\frac{1}{2}(|x_i z_i| + |y_i o|)}$$

From the asymptotic  $PT_{-1}$  inequality, we have

$$e^{\frac{1}{2}(|x_i z_i| + |y_i o|)} \leq e^{\frac{1}{2}(|y_i z_i| + |o x_i|)} + e^{\frac{1}{2}(|x_i y_i| + |o z_i|)} + \delta e^{\frac{1}{2}\rho_i}$$

where  $\rho_i = \max\{|o x_i|, |o y_i|, |o z_i|, |x_i y_i|, |x_i z_i|, |y_i z_i|\}$ . Thus

$$e^{-(x|z)_o} \leq \lim_{i \rightarrow \infty} e^{\frac{1}{2}(|x_i y_i| - |o x_i| - |o y_i|)} + \lim_{i \rightarrow \infty} e^{\frac{1}{2}(|y_i z_i| - |o y_i| - |o z_i|)} + \lim_{i \rightarrow \infty} E_i,$$

where  $E_i = \delta e^{\frac{1}{2}(\rho_i - |o x_i| - |o y_i| - |o z_i|)}$ . Again we easily check that  $E_i \rightarrow 0$  and we obtain in the limit the triangle inequality for  $\rho_o$ .

We use the similar argument to show that  $\rho_o$  satisfies the ptolemaic inequality i.e.

$$e^{-(x|z)_o - (y|w)_o} \leq e^{-(x|y)_o - (z|w)_o} + e^{-(y|z)_o - (x|w)_o}.$$

Choose sequences  $(x_i) \in x, (y_i) \in y, (z_i) \in z, (w_i) \in w$ . Since we have

$$\begin{aligned} e^{-(x_i|z_i)_o - (y_i|w_i)_o} &= e^{-\frac{1}{2}(|x_i o| + |y_i o| + |z_i o| + |w_i o|)} e^{\frac{1}{2}(|x_i z_i| + |y_i w_i|)} \\ &\leq e^{-\frac{1}{2}(|x_i o| + |y_i o| + |z_i o| + |w_i o|)} (e^{\frac{1}{2}(|x_i y_i| + |z_i w_i|)} \\ &\quad + e^{\frac{1}{2}(|y_i z_i| + |x_i w_i|)} + \delta e^{\frac{1}{2}\rho_i}) \\ &= e^{-(x_i|y_i)_o - (z_i|w_i)_o} + e^{-(y_i|z_i)_o - (x_i|w_i)_o} + E_i, \end{aligned}$$

where  $\rho_i = \max\{|x_i y_i|, |x_i z_i|, |x_i w_i|, |y_i z_i|, |y_i w_i|, |z_i w_i|\}$  and

$$E_i = \delta e^{\frac{1}{2}(\rho_i - |x_i o| - |y_i o| - |z_i o| - |w_i o|)}.$$

Again we see that  $E_i \rightarrow 0$  and we obtain in the limit the desired ptolemaic inequality. □

*Remark 3.13.* The above result implies in particular that the asymptotic upper curvature bound (see [BF]) of an asymptotic  $PT_\kappa$  space is bounded above by  $\kappa$ .

## 4 Hyperbolic cones over Möbius spaces

In this chapter we prove Theorem 1.3. Therefore we give (bases on [BoS]) a construction, how to associate to a ptolemaic Möbius space  $(Z, \mathcal{M})$  a hyperbolic space  $X$  (which turns out to be asymptotically  $PT_{-1}$ ), such that  $(Z, \mathcal{M})$  is the canonical Möbius structure of  $\partial_\infty X$ .

Let  $(Z, \mathcal{M})$  be a complete ptolemaic Möbius space. We choose a point  $\omega \in Z$  and an extended metric  $d \in \mathcal{M}$  from the Möbius structure, such that  $\{\omega\} = \Omega(d)$  is the point at infinity. Such a metric exists by Theorem 2.2 and this metric is unique (up to homothety) by Lemma 2.1.

We take now the metric space  $(Z_\omega, d)$ , where  $Z_\omega = Z \setminus \{\omega\}$  and apply the cone construction of [BoS] to it. The space  $\text{Con}(Z_\omega)$  has properties analogous to the hyperbolic convex hull of a set in the boundary of a real hyperbolic space. Set

$$\text{Con}(Z_\omega) = Z_\omega \times (0, \infty).$$

Define  $\rho : \text{Con}(Z_\omega) \times \text{Con}(Z_\omega) \rightarrow [0, \infty)$  by

$$\rho((z, h), (z', h')) = 2 \log \left( \frac{d(z, z') + h \vee h'}{\sqrt{hh'}} \right). \quad (5)$$

It turns out that  $\rho$  satisfies the triangle inequality and is thus a metric, see [BoS]. We write  $|zz'| = d(z, z')$  for  $z, z' \in Z_\omega$ .

**Proposition 4.1.**  *$(\text{Con}(Z_\omega), \rho)$  is asymptotically  $PT_{-1}$ .*

*Proof.* Given arbitrary four points  $x_i = (z_i, h_i) \in \text{Con}((Z_\omega, d))$ ,  $z_i \in (Z_\omega, d)$ ,  $i = 1, 2, 3, 4$ , we have

$$e^{\frac{\rho(x_i, x_j)}{2}} = \frac{|z_i z_j| + h_i \vee h_j}{\sqrt{h_i h_j}}, \quad i \neq j.$$

i.e.

$$|z_i z_j| = \sqrt{h_i h_j} e^{\frac{\rho(x_i, x_j)}{2}} - h_i \vee h_j, \quad i \neq j. \quad (6)$$

Since  $(Z, \mathcal{M})$  is a complete ptolemaic Möbius space,  $(Z_\omega, d)$  is a complete metric space which satisfies the  $PT_0$  inequality, hence we obtain

$$|z_1 z_2| |z_3 z_4| + |z_1 z_4| |z_2 z_3| \geq |z_1 z_3| |z_2 z_4|.$$

Replacing  $|z_i z_j|$  by (6), we have the following inequality

$$\begin{aligned} & (\sqrt{h_1 h_2} e^{\frac{\rho(x_1, x_2)}{2}} - h_1 \vee h_2) (\sqrt{h_3 h_4} e^{\frac{\rho(x_3, x_4)}{2}} - h_3 \vee h_4) \\ & + (\sqrt{h_1 h_4} e^{\frac{\rho(x_1, x_4)}{2}} - h_1 \vee h_4) (\sqrt{h_2 h_3} e^{\frac{\rho(x_2, x_3)}{2}} - h_2 \vee h_3) \\ & \geq (\sqrt{h_1 h_3} e^{\frac{\rho(x_1, x_3)}{2}} - h_1 \vee h_3) (\sqrt{h_2 h_4} e^{\frac{\rho(x_2, x_4)}{2}} - h_2 \vee h_4). \end{aligned}$$

This can be written as

$$\begin{aligned}
& \sqrt{h_1 h_2 h_3 h_4} (e^{\frac{\rho(x_1, x_2)}{2} + \frac{\rho(x_3, x_4)}{2}} + e^{\frac{\rho(x_1, x_4)}{2} + \frac{\rho(x_2, x_3)}{2}} - e^{\frac{\rho(x_1, x_3)}{2} + \frac{\rho(x_2, x_4)}{2}}) \\
& - \sqrt{h_1 h_2} (h_3 \vee h_4) e^{\frac{\rho(x_1, x_2)}{2}} - \sqrt{h_3 h_4} (h_1 \vee h_2) e^{\frac{\rho(x_3, x_4)}{2}} - \sqrt{h_1 h_4} (h_2 \vee h_3) e^{\frac{\rho(x_1, x_4)}{2}} \\
& - \sqrt{h_2 h_3} (h_1 \vee h_4) e^{\frac{\rho(x_2, x_3)}{2}} + \sqrt{h_1 h_3} (h_2 \vee h_4) e^{\frac{\rho(x_1, x_3)}{2}} + \sqrt{h_2 h_4} (h_1 \vee h_3) e^{\frac{\rho(x_2, x_4)}{2}} \\
& + (h_1 \vee h_2)(h_3 \vee h_4) + (h_1 \vee h_4)(h_2 \vee h_3) - (h_1 \vee h_3)(h_2 \vee h_4) \geq 0.
\end{aligned}$$

Using again (6) we obtain

$$\begin{aligned}
& e^{\frac{\rho(x_1, x_2)}{2} + \frac{\rho(x_3, x_4)}{2}} + e^{\frac{\rho(x_1, x_4)}{2} + \frac{\rho(x_2, x_3)}{2}} - e^{\frac{\rho(x_1, x_3)}{2} + \frac{\rho(x_2, x_4)}{2}} \\
& \geq \frac{(h_3 \vee h_4)|z_1 z_2| + (h_1 \vee h_2)|z_3 z_4| + (h_2 \vee h_3)|z_1 z_4| + (h_1 \vee h_4)|z_2 z_3|}{\sqrt{h_1 h_2 h_3 h_4}} \\
& \quad - \frac{(h_2 \vee h_4)|z_1 z_3| + (h_1 \vee h_3)|z_2 z_4|}{\sqrt{h_1 h_2 h_3 h_4}} \\
& \quad + \frac{(h_1 \vee h_2)(h_3 \vee h_4) + (h_1 \vee h_4)(h_2 \vee h_3) - (h_1 \vee h_3)(h_2 \vee h_4)}{\sqrt{h_1 h_2 h_3 h_4}} \quad (7)
\end{aligned}$$

Since  $(a \vee b)(c \vee d) = ac \vee ad \vee bc \vee bd$ ,  $a, b, c, d \in \mathbb{R}$ , we easily obtain that

$$(h_1 \vee h_2)(h_3 \vee h_4) + (h_1 \vee h_4)(h_2 \vee h_3) \geq (h_1 \vee h_3)(h_2 \vee h_4)$$

which shows that the last term in (7) is nonnegative and can be omitted.

We use below that

$$(h_i \vee h_j) + \sqrt{h_i h_j} \geq h_i + h_j$$

for all  $h_i, h_j \geq 0$ .

Let  $\rho = \max_{i,j} \rho_{i,j}$ . Then again by (6)  $|z_i z_j| \leq \sqrt{h_i h_j} e^{\frac{1}{2}\rho}$  and thus

$$\begin{aligned}
& e^{\frac{\rho(x_1, x_2)}{2} + \frac{\rho(x_3, x_4)}{2}} + e^{\frac{\rho(x_1, x_4)}{2} + \frac{\rho(x_2, x_3)}{2}} - e^{\frac{\rho(x_1, x_3)}{2} + \frac{\rho(x_2, x_4)}{2}} + 4e^{\frac{1}{2}\rho} \\
& \geq \frac{(h_3 \vee h_4)|z_1 z_2| + (h_1 \vee h_2)|z_3 z_4| + (h_2 \vee h_3)|z_1 z_4| + (h_1 \vee h_4)|z_2 z_3|}{\sqrt{h_1 h_2 h_3 h_4}} \\
& \quad - \frac{(h_2 \vee h_4)|z_1 z_3| + (h_1 \vee h_3)|z_2 z_4|}{\sqrt{h_1 h_2 h_3 h_4}} + \frac{|z_1 z_2|}{\sqrt{h_1 h_2}} + \frac{|z_3 z_4|}{\sqrt{h_3 h_4}} + \frac{|z_1 z_4|}{\sqrt{h_1 h_4}} + \frac{|z_2 z_3|}{\sqrt{h_2 h_3}} \\
& \geq \frac{(h_3 + h_4)|z_1 z_2| + (h_1 + h_2)|z_3 z_4| + (h_2 + h_3)|z_1 z_4| + (h_1 + h_4)|z_2 z_3|}{\sqrt{h_1 h_2 h_3 h_4}} \\
& \quad - \frac{(h_2 \vee h_4)|z_1 z_3| + (h_1 \vee h_3)|z_2 z_4|}{\sqrt{h_1 h_2 h_3 h_4}} \geq 0
\end{aligned}$$

Therefore  $X$  is asymptotic  $PT_{-1}$ .  $\square$

To finish the proof of Theorem 1.3, we have to show that  $\partial_\infty X$  can be canonically identified with  $Z$ .

We chose a base point  $z_0 \in Z_\omega$  and then  $o := (z_0, 1)$  as base point of  $X$ . We define for simplicity  $|z| := |zz_0|$ . For  $x = (z, h)$  and  $x' = (z', h')$  in  $X$  we compute

$$(x|x')_o = \log\left(\frac{(|z| + h \vee 1)(|z'| + h' \vee 1)}{|zz'| + h \vee h'}\right). \quad (8)$$

**Lemma 4.2.** *A sequence  $x_i = (z_i, h_i)$  in  $X$  converges at infinity, if and only if one of the following holds*

1.  $(z_i)$  is a Cauchy sequence in  $Z_\omega$  and  $h_i \rightarrow 0$ .
2.  $(|z_i| + h_i) \rightarrow \infty$ .

*Proof.* We show first the *if* implication:

Assume 1. that  $(z_i)$  is a Cauchy sequence and  $h_i \rightarrow 0$ . Then equation (8) immediately implies that  $\lim_{i,j \rightarrow \infty} (x_i|x_j)_o = \infty$ .

Assume 2. that  $(|z_i| + h_i) \rightarrow \infty$ . For given  $i, j$  let

$$M_{i,j} = \max\{(|z_i| + h_i \vee 1), (|z_j| + h_j \vee 1)\},$$

$$m_{i,j} = \min\{(|z_i| + h_i \vee 1), (|z_j| + h_j \vee 1)\}.$$

One easily sees

$$M_{i,j} \geq \frac{1}{4}(|z_i z_j| + h_i \vee h_j)$$

thus

$$(x_i|x_j)_o = \log\left(\frac{m_{i,j} M_{i,j}}{|z_i z_j| + h_i \vee h_j}\right) \geq \log\left(\frac{1}{4} m_{i,j}\right)$$

and hence  $\lim_{i,j \rightarrow \infty} (x_i|x_j)_o = \infty$ .

For the *only if* part assume that we have given a sequence  $x_i = (z_i, h_i)$  with  $\lim_{i,j \rightarrow \infty} (x_i|x_j)_o = \infty$ .

We first show that there cannot exist two subsequences  $(x_{i_k})$  and  $(x_{i_l})$  of  $(x_i)$ , such that  $|z_{i_k}| + h_{i_k} \rightarrow \infty$  for  $k \rightarrow \infty$  and  $|z_{i_l}| + h_{i_l} \leq M$  for all  $l$ . If to the contrary such sequences would exist, then we easily obtain using triangle inequalities that

$$|z_{i_k}| + h_{i_k} \vee 1 - 2M - 1 \leq |z_{i_k} z_{i_l}| + h_{i_k} \vee h_{i_l} \leq |z_{i_k}| + h_{i_k} \vee 1 + 2M + 1$$

and hence  $\limsup (x_{i_k}|x_{i_l})_o$  is finite, a contradiction.

Thus either  $|z_i| + h_i \rightarrow \infty$  and we are in case 2 or  $|z_i| + h_i$  is bounded. The boundedness and  $(x_i|x_j)_o \rightarrow \infty$  implies  $\log(|z_i z_j| + h_i \vee h_j) \rightarrow \infty$  and hence  $(z_i)$  is a Cauchy sequence and  $h_i \rightarrow 0$ .  $\square$

**Lemma 4.3.** *One can identify  $Z$  with  $\partial_\infty X$  in a canonical way.*

*Proof.* We define a map  $\chi : Z \rightarrow \partial_\infty X$  by  $z \mapsto [(z, \frac{1}{i})]$  for  $z \in Z_\omega$  and  $\omega \mapsto [(z_0, i)]$ ; here  $[ ]$  denotes the equivalence class of the corresponding sequences. Formula (8) shows that this map is injective. Let now  $\xi \in \partial_\infty X$  be given and be represented by a sequence  $x_i = (z_i, h_i)$ . If  $|z_i| + h_i \rightarrow \infty$  then

$(x_i|(z_0, i))_o \rightarrow \infty$  and  $\xi = \chi(\omega)$ . If  $h_i \rightarrow 0$  and  $(z_i)$  a Cauchy sequence in  $Z_\omega$ , then the  $z := \lim z_i$  exists, since  $(Z, \mathcal{M})$  is a complete Möbius structure. One easily checks  $\xi = \chi(z)$ .  $\square$

**Lemma 4.4.** *The canonical Möbius structure of  $\partial_\infty X$  equals to  $\mathcal{M}$ .*

*Proof.* We consider on  $\partial_\infty X$  the canonical Möbius structure which is given by the metric  $\rho_o(z, z') = e^{-(z|z')_o}$ . Using metric involution we consider the extended metric in the same Möbius class with  $\omega$  as infinitely remote point. This metric is given for  $z, z' \in Z_\omega$  by

$$\rho_{\omega,o}(z, z') = \frac{\rho_o(z, z')}{\rho_o(\omega, z)\rho_o(\omega, z')} = e^{-(z|z')_{\omega,o}}.$$

Now

$$(z|z')_{\omega,o} = (z|z')_o - (\omega|z)_o - (\omega|z')_o.$$

By formula (8) we have

$$(\omega|z)_o = \lim_{i \rightarrow \infty} \log\left(\frac{i(|z_i| + 1)}{|z_i| + i}\right) = \log(|z| + 1)$$

and in the same way  $(\omega|z')_o = \log(|z'| + 1)$ . Using formula (8) we see that for  $z, z' \in Z_\omega$

$$(z|z')_o = \log\left(\frac{(|z| + 1)(|z'| + 1)}{|zz'|}\right).$$

Now we easily compute

$$(z|z')_{\omega,o} = -\log(|zz'|),$$

and hence

$$\rho_{\omega,o}(z, z') = |zz'|.$$

$\square$

## 5 Proof of Theorem 1.4

Our proof relies on the following result, which is a combination of Proposition 4.1 and Theorem [BoS, Theorem 8.2]. For the notion of a visual Gromov hyperbolic space we also refer to that paper or [BS]. Two space  $(X, d_X)$  and  $(Y, d_Y)$  are rough isometric, if there exists  $f : X \rightarrow Y$  and a constant  $K \geq 0$  such that for all  $x, y \in X$

$$|d_X(x, y) - d_Y(f(x), f(y))| \leq K$$

and in addition  $\sup_{y \in Y} d_Y(y, f(X)) \leq K$ .

**Proposition 5.1.** *Assume that  $X$  is a visual Gromov hyperbolic space such that  $e^{-(\cdot|\cdot)_o}$  is bilipschitz to a ptolemaic metric  $d$  on  $\partial_\infty X$ , the  $X$  is rough isometric to an asymptotically  $\text{PT}_{-1}$  space.*

*Proof.* Consider the truncated Cone  $\text{Con}^T(\partial_\infty X, d)$ , which is defined as  $\text{Con}^T(\partial_\infty X) = \partial_\infty X \times (0, D]$ , where  $D = \text{diam}(\partial_\infty X, d)$  again with the metric defined by (5). This is the cone considered in [BoS], where it is shown that  $X$  is rough isometric to  $\text{Con}^T(\partial_\infty X, d)$ . Since by Proposition 4.1 the cone is  $\text{PT}_{-1}$ , the result follows.  $\square$

Now we can finish the proof of Theorem 1.4. We start with some visual Gromov hyperbolic space  $(X, d_X)$  with some base point  $o \in X$ . There exists some  $\varepsilon > 0$ , such that the function  $e^{-\varepsilon(\cdot|\cdot)_o}$  is bilipschitz to a metric  $\rho(\cdot, \cdot)$  on  $\partial_\infty X$  (see e.g. [BS, Theorem 2.2.7]). By a result of Lytchak (see [FS1, Proposition 8])  $\rho^{\frac{1}{2}}$  is a ptolemaic. Clearly  $e^{-\frac{\varepsilon}{2}(\cdot|\cdot)_o}$  is bilipschitz to the metric  $\rho^{\frac{1}{2}}(\cdot, \cdot)$ . Thus the visual Gromov hyperbolic space  $(X, \frac{\varepsilon}{2}d_X)$  satisfies the assumptions of Proposition 5.1 and is rough isometric to an asymptotically  $\text{PT}_{-1}$  space. Hence  $(X, d_X)$  is rough similar to this space.

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